Homework 1, due 9/3

1. For each $c \in D(0,1)$ define the transformation L_c by
$$L_c(z) = \frac{z-c}{1-\bar{c}z}.$$
Prove that L_c maps the unit disk $D(0,1)$ onto the unit disk, and the unit circle $S^1 = \{z : |z| = 1\}$ onto the unit circle.

2. If $f : \Omega \to \mathbf{C}$ is holomorphic on a connected open set $\Omega \subset \mathbf{C}$, prove the following:
   (i) If $f'(z) = 0$ for all $z \in \Omega$, then f is constant.
   (ii) If there exists $c \in \mathbf{C}$ such that $f(z) = c \cdot \overline{f(z)}$ for every $z \in \Omega$, then f is constant.
   (iii) If $f(\Omega) \subset \mathbf{R}$, then f is constant.

3. Suppose that $f : \Omega \to \mathbf{C}$ has continuous first partial derivatives, and the real and imaginary parts of f satisfy the Cauchy-Riemann equations (so in particular f is holomorphic). Let γ be the boundary of a smooth domain in Ω oriented positively. Show, using Green's Theorem in the plane, that
$$\int_\gamma f(z)dz = 0.$$

4. Let $|a| < r < |b|$. Show that
$$\int_\gamma \frac{1}{(z-a)(z-b)}\, dz = \frac{2\pi i}{a-b},$$
where γ is the circle of radius r centered at the origin (oriented counter-clockwise).